How the Binomial, Poisson and Normal Distributions are related to each other.

Binomial : $P_B(r|n) = \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r}$ mean, $\mu = np$; variance, $\sigma^2 = np(1-$; std. dev. σ

Poisson : $P_P(r) = \frac{\mu^r e^{-\mu}}{r!}$ mean : μ; variance : μ.

Normal : $p(x)dx = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$ mean : μ; std. dev : σ

Binomial : Probability of obtaining r "successes" in n "trials", where p is the probability of obtaining success in a single trial. (ie tossing a coin)

h = heads
t = tails

eg : $\underbrace{\text{htttthhthhththhhhhtthhhhthtthttt}}$

- probability of this precise configuration : $p^r(1-p)^{n-r}$
- number of ways of ordering the r heads and $(n-r)$ tails : $\frac{n!}{r!(n-r)!}$

This becomes the Poisson dist'n in the limit of
$n \rightarrow \infty$
$p \rightarrow 0$
$np \rightarrow$ const (still μ)

ie replace all the heads except for a very tiny number with tails and increase n (ie "zoom out") to get :

$$\downarrow \qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \quad \downarrow$$

htttttttttt............tttttttthtttttt.............h.....h........

ie incredibly unlikely that we obtain success but we try so many times that we still get, on average, μ successes.

think: lightning strikes - how many lightning strikes in the next 30 mins given that on average we get ten every hour (so that $\mu = 5$)

or : how many bad apples in this crate of 50 apples given that on average, 10% are bad ($\Rightarrow \mu = 5$)

answer : $\quad P(r \text{ bad apples}) = \dfrac{5^r}{r!} e^{-5}$

So : Binomial $\rightarrow$ Poisson as follows :

write p as $\quad p = \dfrac{\mu}{n}$

$$\therefore \quad P_B(r|n) = \frac{n!}{r!(n-r)!} \left(\frac{\mu}{n}\right)^r \left(1 - \frac{\mu}{n}\right)^{n-r}$$

and $\quad \dfrac{n!}{(n-r)!} = n(n-1)(n-2)(n-3)\ldots\ldots(n-(r+1))$

but $n \gg r$ so get $\quad \underbrace{n.n \; n \; n \ldots\ldots n}_{r} = n^r$

$$\text{and} \quad \left(1 - \frac{\mu}{n}\right)^{n-r} \simeq \left(1 - \frac{\mu}{n}\right)^n$$

$$= 1 + n\left(-\frac{\mu}{n}\right) + \frac{n(n-1)}{2}\left(-\frac{\mu}{n}\right)^2 + \frac{n(n-1)(n-2)}{2.3}\left(-\frac{\mu}{n}\right)^3 + \ldots\ldots$$

$$\simeq 1 + \cancel{n}\left(\frac{-\mu}{\cancel{n}}\right) + \frac{\cancel{n^2}}{2}\left(\frac{-\mu}{\cancel{n}}\right)^2 + \frac{\cancel{n^3}}{3!}\left(\frac{-\mu}{\cancel{n}}\right)^3 + \dots$$

$$= 1 + (-\mu) + \frac{(-\mu)^2}{2!} + \frac{(-\mu)^3}{3!} + \dots$$

$$= e^{-\mu} .$$

So putting it back together gives:

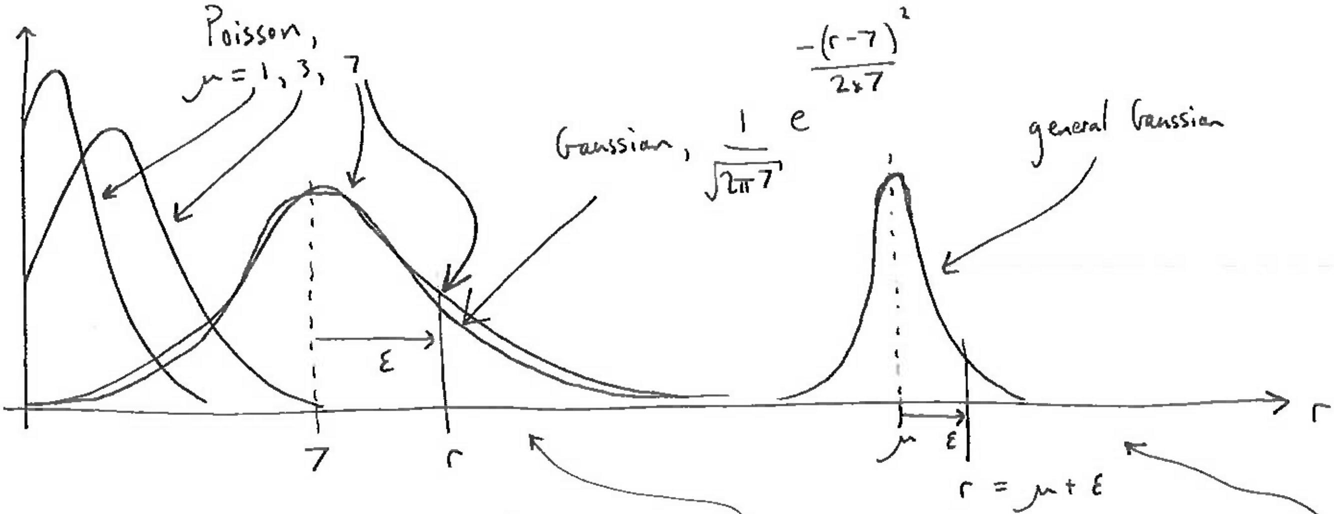
$$\lim_{\substack{n \to \infty \\ p \to 0 \\ np = \mu}} P_B(r|n) = \frac{\cancel{n^r}}{r!}\left(\frac{\mu}{\cancel{n}}\right)^r e^{-\mu}$$

$$= \frac{\mu^r}{r!} e^{-\mu} \qquad \text{ie } \underline{\text{Poisson Dist'n}}$$

and $$\sum_{r=0}^{\infty} \frac{\mu^r}{r!} e^{-\mu} = e^{-\mu} \underbrace{\sum_{r=0}^{\infty} \frac{\mu^r}{r!}}_{e^{\mu}} = 1 \quad \checkmark\checkmark .$$

so it is still normalised, as expected.

The Poisson dist'n becomes the Normal (ie Gaussian) dist'n in the limit of $r \to \infty$.



In fact, in this limit, the Poisson dist'n becomes a Gaussian with variance = mean

– we will then relax that condition

$$P_p(r) = e^{-\mu} \cdot \frac{\mu^r}{r!} \qquad \text{(Poisson)}$$

Consider $\log_e P_p = r \log \mu - \mu - \log r!$

and use Stirling's approximation $\log r! = r \log r - r - \frac{1}{2} \log 2\pi r$

which is valid for large r.

then $\log P_p = r - \mu - r \log \frac{r}{\mu} - \frac{1}{2} \log 2\pi r$

now: both r and $r \log r$ rise faster than $\log r$, so we need to treat the first terms $\left(r - \mu - r \log \frac{r}{\mu}\right)$ more accurately than the last term $-\frac{1}{2} \log 2\pi r$. Let's expand the former to second order in $\varepsilon = r - \mu$ and to zero'th order in the latter :

$= r - \sigma^2$

ie let $\log 2\pi r \simeq \log 2\pi\mu$

$= \log 2\pi\sigma^2$

variance = mean, for Poisson dist'n

So $\log P_p = \underbrace{\varepsilon - (\sigma^2 + \varepsilon) \log\left(1 + \frac{\varepsilon}{\sigma^2}\right)}_{\text{expand to second order in } \varepsilon} - \frac{1}{2} \log 2\pi\sigma^2$

$$\log\left(1 + \frac{\varepsilon}{\sigma^2}\right) = \frac{\varepsilon}{\sigma^2} - \frac{1}{2}\left(\frac{\varepsilon}{\sigma^2}\right)^2 + \ldots$$

$$\therefore \log P_p = \varepsilon - (\sigma^2 + \varepsilon)\left(\frac{\varepsilon}{\sigma^2} - \frac{1}{2}\left(\frac{\varepsilon}{\sigma^2}\right)^2\right) - \frac{1}{2} \log 2\pi\sigma^2$$

$$= \cancel{\varepsilon - \varepsilon} \underbrace{- \frac{\varepsilon^2}{\sigma^2} + \frac{1}{2} \frac{\varepsilon^2}{\sigma^2}}_{-\frac{\varepsilon^2}{2\sigma^2}} - \frac{1}{2} \log 2\pi\sigma^2$$

So $\log P_p = \log \dfrac{1}{\sqrt{2\pi\sigma^2}} - \dfrac{\varepsilon^2}{2\sigma^2}$

$$\therefore \lim_{r \to \infty} P_p(r) = p_c(r) = \underline{\underline{\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\frac{-(r-\mu)^2}{2\sigma^2}}}}$$ ie <u>Normal Distribution.</u>

So far we have set $\sigma^2 = \mu$, but we can now relax that assumption to get a general Normal distribution.

So, in summary: